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LETTER TO THE EDITOR

Supersymmetric manipulation of quasienergy states: application to the geometric phase

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Abstract

Time-dependent supersymmetry allows one to delete quasienergy levels for time-periodic Hamiltonians and to create new ones. We illustrate this by examining an exactly solvable model related to the simple harmonic oscillator with a time-varying frequency. For a nonharmonic example we present the change of the geometric phase due to a supersymmetry transformation.

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One may say without exaggeration that non-relativistic quantum mechanics is essential to understanding the basic laws of nature. Thus, the Schrödinger equation is one of the most fundamental constructs of modern natural science and over the past century, as evidenced by a vast literature, great progress has been made in developing various consequences of the *time-independent* Schrödinger equation describing stationary processes. In the real world, however, nearly everything changes with time, and stationary processes are exceptional (perhaps even impossible), but there is ample evidence for believing that there will soon be similar advances in the study of non-stationary processes and the consequences of the *time-dependent* Schrödinger equation. The end of the last century saw some progress in this area: an extensive investigation into the properties of *time-periodic* Hamiltonians, their dynamical invariants [1] and symmetries [2], dynamical pulsed coherent states [3], Floquet quanta [4] and application in optics [5]. An area that has been attracting interest recently is calculating the various classes of geometric phases [6] (for example Berry's adiabatic phase [7] and its generalization, the Aharonov–Anandan phase [8]) for time-dependent Hamiltonians, although so far there is a shortage of exact calculations for realistic physical models, i.e. away from the adiabatic limit (in this respect, see [9]). Indeed, the system treated in most detail is the exactly solvable harmonic oscillator with time-dependent mass and/or frequency [10].

It has long been known how to construct new exactly solvable time-independent Hamiltonians from an initial one by means of the Darboux, or supersymmetry, transformation

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(see e.g. [11, 12]), and the procedure has recently been generalized to include time dependence [12, 13]. Although stationary state energies do not exist for a time-dependent Hamiltonian, if the time dependence is periodic, *quasienergies* play a similar role [14, 15]. In this letter, based on the simple harmonic oscillator model, having a time-periodic frequency, we show that this analogy is much deeper than previously recognized and that it can be extended to the level of supersymmetry. In particular, we show that processes such as the creation and annihilation of quasienergies may be effected easily by time-dependent supersymmetry (or Darboux) transformations.

Since exact eigenstates are known for these new time-dependent Hamiltonians, their remarkable properties, such as the geometric phases, can be investigated in detail. We demonstrate here that the Aharonov–Anandan geometric phase, for systems generated from our model harmonic oscillator Hamiltonian, simply acquires an additive correction under a Darboux transformation. As is well known, the Aharonov–Anandan geometric phase appears as the holonomy in the projective state space (the structure involved here is a Hilbert bundle with base the projective state space and fibre $U(1)$), and it is related to the exact (nonadiabatic) evolution of any cyclic state (a closed path in the projective Hilbert space) in a time-dependent Hamiltonian.

We begin with the Schrödinger equation for the following one-dimensional time-dependent Hamiltonian

$$h_0 = -\partial_x^2 + \omega^2(t)x^2 \qquad \omega(t+T) = \omega(t) \tag{1}$$

where $\partial_x \equiv \partial/\partial_x$ and appropriate units have been chosen. A complete set of solutions normalized to unity on the real line is [16]

$$\psi_n(x,t) = (\bar{\varepsilon}(t)/\varepsilon(t))^{n/2+1/4} \exp(\mathrm{i}\dot{\gamma}(t)z^2(x,t))\, f_n(x,t) \tag{2}$$

where

$$f_n(x,t) = \frac{N_n}{[\gamma(t)]^{1/4}} H_n(z)\, \mathrm{e}^{-z^2/2} \qquad z(x,t) = x/\sqrt{8\gamma(t)}. \tag{3}$$

Here $N_n = (2^{n+1}n!\sqrt{2\pi})^{-1/2}$, $\gamma(t) = \varepsilon(t)\bar{\varepsilon}(t)$, $H_n(z)$ is a Hermite polynomial and $\varepsilon(t)$, $\bar{\varepsilon}(t)$ are two linearly independent solutions of the classical equation of motion for the harmonic oscillator

$$\ddot{\varepsilon}(t) + 4\omega^2(t)\varepsilon(t) = 0. \tag{4}$$

A dot over a symbol represents the derivative with respect to time and we choose these solutions so that their Wronskian $\dot{\varepsilon}(t)\bar{\varepsilon}(t) - \varepsilon(t)\dot{\bar{\varepsilon}}(t) = \frac{\mathrm{i}}{2}$. In general, the solutions of (4) are of three types: (i) both $\varepsilon(t)$ and $\bar{\varepsilon}(t)$ are stable, (ii) they are both unstable, (iii) one is stable, the other is unstable. It is known [15] that only in the first case do the functions (2) have the property

$$\psi_n(x, t+T) = \mathrm{e}^{-\mathrm{i}E_nT}\psi_n(x,t) \tag{5}$$

where $E_n = \left(n + \frac{1}{2}\right)\delta$ is called the *quasienergy* [14] and δ is defined by the equation

$$\varepsilon(t+T) = \varepsilon(t)\,\mathrm{e}^{\mathrm{i}\delta T}. \tag{6}$$

According to Floquet's theorem a (complex) solution $\varepsilon(t)$ to equation (4) satisfying (6) can always be found in case (i). We choose the complex conjugate of $\varepsilon(t)$ to be the second, linearly independent, solution to (4). Here and below a bar over a quantity denotes its complex conjugate and we note that $\gamma(t)$ is real. It is clear from (5) that quasienergies E_n are defined modulo 2π and if they are commensurable with $\frac{2\pi}{T}$, the functions ψ_n are periodic.

A further useful property of the functions (2) is that there exist ladder operators

$$a = \varepsilon(t)\partial_x - \mathrm{i}\dot{\varepsilon}(t)x/2 \qquad a^+ = -\bar{\varepsilon}(t)\partial_x + \mathrm{i}\dot{\bar{\varepsilon}}(t)x/2 \tag{7}$$

such that $a\psi_n = \frac{\sqrt{n}}{2}\psi_{n-1}$, $a^+\psi_n = \frac{\sqrt{n+1}}{2}\psi_{n+1}$. It follows from here that they are eigenstates of the symmetry operator $G = (aa^+ + a^+a)/2$, $G\psi_n = \frac{1}{8}(2n+1)\psi_n$.

Below we shall apply the approach presented in [13] for constructing new exactly solvable time-dependent Hamiltonians having the same quasienergy spectrum, with the possible exception of a few levels, as a given one. The procedure is simply the time-dependent generalization of the usual supersymmetry construction [11]. A new Hamiltonian h_1 is defined by means of a nodeless solution $u(x,t)$, called the *transformation function*, to the initial Schrödinger equation, $\mathrm{i}\partial_t u(x,t) = h_0 u(x,t)$, subject to the additional condition $\partial_x^3(\ln u/\bar{u}) = 0$, which guarantees that h_1 has the form

$$h_1 = h_0 - \partial_x^2 \ln|u(x,t)|^2 \tag{8}$$

the potential being a real function. Solutions of the corresponding Schrödinger equation are obtained by applying the differential operator

$$L = L_1(t)[-\partial_x + (\partial_x \ln u(x,t))] \tag{9}$$

to ψ_n:

$$\varphi_n(x,t) = M_n^{-1/2} L\psi_n(x,t) \tag{10}$$

where the time-independent factor $M_n^{-1/2}$ guarantees that the functions (10) are normalized to unity. The operator L is defined in terms of the same function $u(x,t)$ as in (8) and the function

$$L_1(t) = \exp\left[2\int \mathrm{Im}\left(\partial_x^2 \ln u\right)\mathrm{d}t\right] \tag{11}$$

which depends only on t [13]. The operator

$$L^+ = L_1[\partial_x + (\partial_x \ln \bar{u}(x,t))] \tag{12}$$

adjoint to L, realizes the transformation in the opposite direction, from solutions of the Schrödinger equation with Hamiltonian h_1 to those of the Hamiltonian h_0. Thus, the superposition L^+L is a symmetry operator for the initial Schrödinger equation and the function $u(x,t)$ is an eigenfunction of this operator. The normalization factor in (10) is equal to its mean value $M_n = \langle\psi_n|L^+L|\psi_n\rangle$.

Among the functions (2), only ψ_0 is nodeless and suitable for using as a transformation function; it produces only a shift of the Hamiltonian by an x-independent value. This is a manifestation of the well-known shape-invariance property [11] in the time-dependent case. Any other function $u_k(x,t) = \psi_k(x,t), k > 0$ taken as the transformation function will produce a potential with k poles corresponding to the zeros of $\psi_k(x,t)$, which clearly has no physical meaning if the variable x runs over the whole real line. Nevertheless, the transformed Hamiltonian h_1 can be taken as the initial one for the next transformation step and if this is realized with the transformation function $\tilde{u}_{k+1}(x,t) = Lu_{k+1}(x,t)$, all the poles are removed and the resulting Hamiltonian $h_2^{(k)} = h_0 - A_2^{(k)}(x,t)$ is physically admissible. For the potential difference one gets

$$A_2^{(k)}(x,t) = \frac{1}{4\gamma(t)}\left[\frac{\mathcal{S}_k''(z)}{\mathcal{S}_k(z)} - \left(\frac{\mathcal{S}_k'(z)}{\mathcal{S}_k(z)}\right)^2 - 2\right]. \tag{13}$$

Here $\mathcal{S}_k(z) = \sum_{j=0}^{k}(k!/2^j j!)H_j^2(z)$ and for the first three of these functions one has the simple expressions: $\mathcal{S}_0(z) = 1$, $\mathcal{S}_1(z) = 2z^2 + 1$, $\mathcal{S}_2(z) = 4z^4 + 3$. The Hamiltonian $h_2^{(k)}$ has the same system of quasienergies as h_0 except for those corresponding to $n = k$ and $n = k+1$ which are now deleted. We can repeat this process without any restrictions and get a Hamiltonian

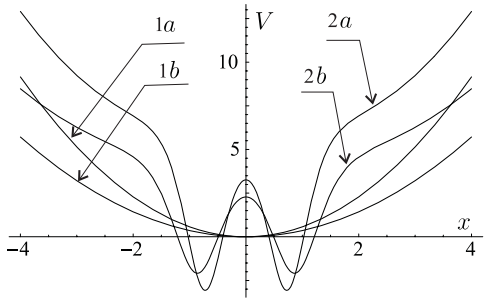

Figure 1. Potentials $V_2^{(2)}(x,t)$ at $t=0$ (curve $2a$) and at $t=T/2$ (curve $2b$) together with the harmonic oscillator potential (curves $1a$ and $1b$, respectively).

having an initial quasienergy spectrum with any number of lacunae composed of two adjacent levels. The potential

$$V_2^{(k)}(x,t)=\omega^2(t)x^2-A_2^{(k)}(x,t) \tag{14}$$

looks like a harmonic oscillator potential, at the bottom of which there are k additional minima. The behaviour of the $k=2$ case, which is typical, is sketched in figure 1 together with the harmonic oscillator potential.

The opposite process, the creation of new quasienergy levels, is also possible. For this purpose we need *unphysical* solutions of the initial Schrödinger equation, which do not belong to the Hilbert space and have growing asymptotic behaviour as $|x|\to\infty$ any fixed time moment. It is not difficult to see that the functions

$$u_k(x,t)=\gamma^{-1/4}(t)(\varepsilon(t)/\bar{\varepsilon}(t))^{k/2+1/4}H_k(\mathrm{i}z)\exp((\mathrm{i}\dot{\gamma}(t)+1/2)z^2) \tag{15}$$

have this property. For any even value $k=2\ell$ these are nodeless and suitable for using as transformation functions. They produce the new Hamiltonians

$$h_1^{(2\ell)}=-\partial_x^2+V_1^{(2\ell)}(x,t)\qquad \ell=0,1,\ldots \tag{16}$$

with potentials

$$V_1^{(2\ell)}(x,t)=\omega^2(t)x^2-A_1^{(2\ell)}(x,t) \tag{17}$$

where

$$A_1^{(2\ell)}(x,t)=\frac{1}{4\gamma(t)}\left[1+4\ell(2\ell-1)\frac{q_{2\ell-2}(z)}{q_{2\ell}(z)}-8\ell^2\left(\frac{q_{2\ell-1}(z)}{q_{2\ell}(z)}\right)^2\right] \tag{18}$$

and $q_k(z)=(-\mathrm{i})^k2^{-k/2}H_k(\mathrm{i}z)$. Using the recursion relation for Hermite polynomials one finds $q_0(z)=1$, $q_1(z)=\sqrt{2}z$, $q_{k+1}(z)=\sqrt{2}zq_k(z)+kq_{k-1}(z)$. The fact that a new quasienergy level is created by this process follows from the property that the function $v_k=1/(L_1(t)\bar{u}_k)$, where $L_1=\sqrt{\gamma(t)}$, is a square integrable solution of the transformed Schrödinger equation. It is easy to see that it corresponds to the quasienergy $E=-\delta\left(k+\frac{1}{2}\right)$, which, in general, is different from all the other quasienergies $E_n=\delta\left(n+\frac{1}{2}\right)$, $n=0,1,\ldots$. The case $k=0$ reproduces the harmonic oscillator Hamiltonian shifted by an x-independent quantity. The first nontrivial case corresponds to $k=2$. We display a typical potential, $V_1^{(2)}(x,t)$, in figure 2.

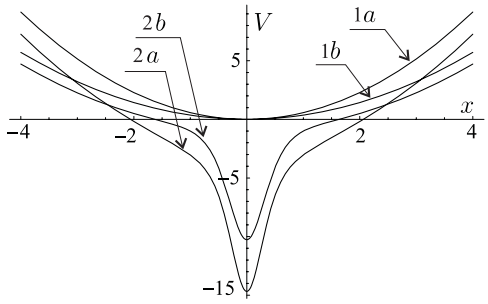

Figure 2. Potential $V_1^{(2)}(x,t)$ at $t=0$ (curve $2a$) and at $t=T/2$ (curve $2b$) together with the harmonic oscillator potential (curves $1a$ and $1b$, respectively).

After applying the operator (9) to the functions (2) one gets solutions (10) of the transformed equation corresponding to the quasienergies E_n. For the case $k=2$ they can easily be expressed in terms of $\psi_n(x,t)$:

$$\varphi_n(x,t)=\frac{1}{\sqrt{n+3}}\left[\sqrt{n+1}\sqrt{\frac{\varepsilon(t)}{\bar{\varepsilon}(t)}}\psi_{n+1}(x,t)+\frac{\sqrt{2}z\psi_n(x,t)}{z^2+1/2}\right]. \tag{19}$$

The normalization factor here is calculated by noting that the symmetry operator L^+L is simply the shifted G operator, $L^+L=G+5/8$.

We now turn to the transformation of the Aharonov–Anandan phase and consider for simplicity the first nontrivial case $k=2$. In these calculations we are using the standard approach (see e.g. [8]). The global phase change ϕ_n of a cyclic solution of the Schrödinger equation during time T is obtained directly from (2) for the harmonic oscillator potential (and from (19) for transformed potentials). The dynamical part of this global phase is given in terms of the mean energy value $\int_0^T\langle\psi_n|\mathrm{i}\dot{\psi}_n\rangle\,\mathrm{d}t$ (or $\int_0^T\langle\varphi_n|\mathrm{i}\dot{\varphi}_n\rangle\,\mathrm{d}t$). The Aharonov–Anandan phase β_n^0 for the functions in (2) is obtained from the overall phase change by adding the dynamical component

$$\beta_n^0=\phi_n+\int_0^T\langle\psi_n|\mathrm{i}\dot{\psi}_n\rangle\,\mathrm{d}t. \tag{20}$$

After some algebra we get the following results:

$$\phi_n=-\left(n+\frac{1}{2}\right)\delta T \tag{21}$$

$$\langle\psi_n|\mathrm{i}\dot{\psi}_n\rangle=\frac{\mathrm{i}}{4}(2n+1)\frac{\mathrm{d}}{\mathrm{d}t}\ln(\bar{\varepsilon}(t)/\varepsilon(t)-\frac{1}{8}\langle\psi_n|x^2|\psi_n\rangle\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\gamma(t) \tag{22}$$

where the mean value of the square coordinate is $\langle\psi_n|x^2|\psi_n\rangle=4(2n+1)\gamma(t)$.

The structure of the functions $\varphi_n(x,t)$ is similar to those in (2) and therefore the average energy is given by (22) with the replacement $\psi_n\to\varphi_n$. To calculate the mean value of the square coordinate for the functions (19) we use the property $L^+L=G+5/8$ and the fact that the functions (2) are eigenfunctions for G to get

$$\langle\varphi_n|x^2|\varphi_n\rangle=\langle\psi_n|x^2|\psi_n\rangle+\frac{4\sqrt{\gamma(t)}}{\sqrt{n+3}}\langle\psi_n|x|\varphi_n\rangle.$$

The first integral here is standard. For the second one we have found the following expression $\langle\psi_n|x|\varphi_n\rangle = 2\sqrt{\gamma(t)/(n+3)}\big[n+3-\mathcal{I}_n\big(\frac{1}{2}\big)\big]$, where

$$\mathcal{I}_n(a) = \frac{\mathcal{T}_n(a)}{\sqrt{\pi}n!2^n} \qquad \mathcal{T}_n(a) = \int_{-\infty}^{\infty} \frac{H_n^2(x)}{x^2+a}\,\mathrm{e}^{-x^2}\,\mathrm{d}x.$$

For the latter integral we obtained the recursion relation

$$\mathcal{T}_n(a) = -2\mathcal{T}_{n-1}(a) + 4(n-1)^2\mathcal{T}_{n-2}(a) - 4a\mathcal{T}'_{n-1}(a)$$

which follows directly from that for the Hermite polynomials. The initial values of this function may be calculated in a straightforward way

$$\mathcal{T}_0(a) = \frac{\pi}{\sqrt{a}}\,\mathrm{e}^a\,\mathrm{erfc}\sqrt{a} \qquad \text{and} \qquad \mathcal{T}_1(a) = 4\sqrt{\pi} - 4a\mathcal{T}_0(a).$$

Therefore, the mean energy in the states (19) is given by

$$\langle\varphi_n|\mathrm{i}\dot{\varphi}_n\rangle = \langle\psi_n|\mathrm{i}\dot{\psi}_n\rangle - \left(1 - \frac{\mathcal{I}_n\big(\frac{1}{2}\big)}{n+3}\right)\gamma(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\gamma(t). \tag{23}$$

After being integrated over a period T the first term on the RHS of (22) gives us exactly the overall phase change ϕ_n, both for the initial states and for the transformed ones (see (23)); the Aharonov–Anandan phase is determined only by the second term. Thus, the geometric phases β_n^0 for all states $\psi_n(x,t)$ are determined by the geometric phase for the ground state:

$$\beta_n^0 = (2n+1)\beta_0^0 \qquad \beta_0^0 = -\frac{1}{2}\int_0^T \gamma(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\gamma(t)\,\mathrm{d}t.$$

For the transformed Aharonov–Anandan phase from (23) one gets

$$\beta_n^1 = \beta_n^0 + 2\left[1 - \frac{\mathcal{I}_n\big(\frac{1}{2}\big)}{n+3}\right]\beta_0^0.$$

Hence, once the quantity β_0^0 is known, we can easily calculate the Aharonov–Anandan phase both for the harmonic oscillator and for the Hamiltonian $h_1^{(2)}$.

As a numerical illustration we have chosen a model for which analytic solutions of equation (4) are available:

$$\omega(t) = \sqrt{\omega_0^2 - \tfrac{1}{2}\wp(t+\omega_i)}. \tag{24}$$

Here ω_0 is a parameter of the model along with the real ($\omega_r = T$) and imaginary (ω_i) periods of the Weierstrass $\wp$ function. If we eliminate the parameter ω_0 in favour of d given by $\wp(d) = -4\omega_0^2$, the solutions of equation (4) have the form [17] $\varepsilon(t) = (\sigma(t+\omega_i+d)/\sigma(t+\omega_i))\exp(-t\zeta(d))$. Here σ and ζ are (non-elliptic) Weierstrass functions. Figures 1 and 2 are plotted with $\omega_r = -\mathrm{i}\omega_i = 2$ and $\omega_0 \cong 0.5978$. We have also found the value $\beta_0^0 = 0.0149$.

We make a further comment concerning the time-dependent supersymmetry underlying our approach. The transformation operators are related to eigenfunctions (not necessarily 'physical') of the symmetry operator G, $Gu_k = g_k u_k$ ($g_k = 5/8$ for our choice) which accounts for the factorization $L^+L = G - g_k$. When the order of transformation operators is interchanged, one gets a symmetry operator $\tilde{G}$ for the transformed Schrödinger equation, $LL^+ = \tilde{G} - g_k$. The operators G and $\tilde{G}$ are supersymmetric partners from which a matrix operator $\mathcal{G} = \mathrm{diag}(G, \tilde{G})$ can be constructed. It acts in the space spanned by the basis vectors

$$\Psi_n^{(1)} = \begin{pmatrix}\psi_n\\ 0\end{pmatrix} \qquad \Psi_n^{(2)} = \begin{pmatrix}0\\ \varphi_n\end{pmatrix}$$

where the functions ψ_n and φ_n are given in (2) and (19), respectively. So, just as in the usual supersymmetric approach, one has a two-fold degenerate spectrum except for the ground-state level which is non-degenerate. This means that we have constructed here a model with unbroken supersymmetry. Finally we note that supercharge operators, closing a superalgebra, can be constructed as usual with the help of the transformation operators L and L^+.

In summary, in this letter we have constructed, in principle, an infinite number of time-periodic Hamiltonians having almost unlimited complexity, for which the Aharonov–Anandan phase can be determined explicitly, and have illustrated the procedure for a Hamiltonian whose time dependence is given by the square root of an elliptic function. This extends greatly the set of previously known cases. We feel that this opens the way for a systematic investigation of the geometric phases for one-dimensional quantum systems exhibiting supersymmetric properties.

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